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## Asset Preference and the Measurement of Expected Utility: Some Problems

*Matt Hassett*

*R. Stephen Sears*

*Gary L. Trennepohl*

College of Commerce and Business Administration  
Bureau of Economic and Business Research  
University of Illinois Urbana-Champaign



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Asset Preference and the Measurement of  
Expected Utility: Some Problems

Matt Hassett  
Arizona State University

R. Stephen Sears, Assistant Professor  
Department of Finance

Gary L. Trennepohl  
University of Missouri at Columbia



## ABSTRACT

This paper shows that serious analytical errors may occur in expected utility theory when Taylor series approximation methods are used without careful attention to underlying mathematical assumptions. Recent studies have developed theory incorporating skewness of return into expected utility calculations based on a Taylor series approximation. It is apparent that this theory is invalid if assumptions for application of a Taylor series cannot be met. Errors may occur if returns fall outside the region of convergence of the utility function or if the partial sums of the Taylor series provide poor approximations to the utility function. Stylized examples are presented to illustrate miscalculation of utility when the various assumptions are violated. These examples are motivated by the new spectrum of financial securities which allow investors to create almost any desired expected return distribution.



## Asset Preference and the Measurement of Expected Utility: Some Problems

Under certain assumptions about the investor's utility function,  $U$ , or a portfolio's return distribution, it is possible to calculate investor utility using the first two or three terms of a Taylor series expansion. However, if these assumptions are violated, deletion of higher moment terms seriously alters calculated utility and may lead to erroneous conclusions about asset preference.

Portfolio theory was developed using quadratic utility functions and return distributions which can be fully characterized by their first two moments (e.g., normal, binomial or uniform). Such utility functions and distributions permit solution for expected utility by a quadratic Taylor series approximation. Deficiencies in using only the first two moments have been noted by Arditti [1], Jean [7], Levy [11], Simkowitz and Beedles [20], Beedles [2], and Kraus and Litzenberger [10] who advocate the importance of skewness for evaluating investor utility. Francis [5] and Friend and Westerfield [6] have presented contrary evidence.

Scott and Horvath [19] developed a mathematical theory of preference for higher distribution moments in spite of the lack of economic interpretation (see Kaplanski [9]). Optimal expected utility based on a return distribution's first three moments has been derived by Conine and Tamarkin [4] and Kane [8] who use a truncated Taylor series of the first three distribution moments to approximate expected utility for stylized utility functions.

Assumptions for the use of a Taylor series to calculate  $E(U)$  are outlined in [2] and recent papers [4,8] have developed theory strongly dependent on these assumptions. It is apparent that the theory is invalid if these assumptions cannot be met. Potential errors can result if: (1) returns,  $R$ , fall outside of the region of convergence of the utility function  $U(R)$  (where the Taylor series of utility function does not coverge) or (2) the quadratic and/or cubic partial sums of the Taylor series provide poor approximations to the utility function. The second error is likely to occur with highly skewed distributions. It can be expected that the controversy about proper specification of expected investor utility will continue because the listed options and futures markets enable investors to create portfolios with distinctive highly skewed expected return distributions [15].

Loistl [13] has warned of problems that may occur if truncated Taylor series approximation methods are used in utility analysis. However, his work deals primarily with the normal and lognormal (continuous) return distributions, and does not analyze the mean-variance-skewness approximations used in current finance literature [4,8].

Our study will focus on the problems inherent in two and three moment Taylor series expected utility<sup>1</sup> approximations and discrete return distributions. Part I describes the traditional Taylor series approximation for  $E(U)$  while part II defines and illustrates problems identified in the use of a Taylor series to determine  $E(U)$ . Part III relates the impact of these problems on the results of recent studies. Part IV presents the implication of this research to future works about asset preference.

# I. Taylor Series Approximation of Utility Functions

Let  $W_0$  be the initial investor wealth,  $R$  a random variable representing relative return on investment and  $U(RW_0)$  a utility function quantifying the utility to an investor of the relative return  $R$  on initial wealth  $W_0$ . Let  $\mu_R$  denote the mean of the random variable  $R$ . If the utility function  $U$  is analytic in a region containing all possible values of the random variable  $R$ , we may expand  $U$  in a Taylor series for any resulting single period wealth  $W_0 R$ . It is most useful to expand around  $\mu_R$ :

$$U(W_0 R) = \sum_{n=0}^{\infty} \frac{U^{(n)}(W_0 \mu_R) (W_0 R - W_0 \mu_R)^n}{n!} \quad (1)$$

For the widely used utility functions  $U(R) = \ln(R)$  and  $U(R) = W_0 R^p$  ( $p$  a fixed number in the interval  $(0,1)$ ), the effect of initial wealth  $W_0$  can be "separated out" and one need only analyze:

$$U(R) = \sum_{n=0}^{\infty} \frac{U^{(n)}(\mu_R)}{n!} (R - \mu_R)^n \quad (2)$$

where  $U^{(n)}(\mu_R)$  denotes the  $n$ th derivative of the utility function at the point  $\mu_R$ .

For simplicity, we restrict our analysis to the "separated" expansion (2). Since the coefficient of  $(R - \mu_R)^n$  is merely a constant,  $a_n$ , the terms can be further simplified by writing:

$$a_n = \frac{U^{(n)}(\mu_R)}{n!} \quad (3)$$

and rewriting (2) as :

$$U(R) = \sum_{n=0}^{\infty} a_n (R - \mu_R)^n \quad (4)$$

# A. Use of Taylor Series Approximation to Find $E(U(R))$

The following sequence of steps is employed in the literature to provide an approximate formula for  $E(U(R))$ .

S1. Take expected values of each side of (1), to obtain:

$$E(U(R)) = \sum_{n=0}^{\infty} a_n E(R - \mu_R)^n \quad (5)$$

Since  $E(R - \mu_R)^n$  is the  $n^{\text{th}}$  central moment of the distribution of  $R$  (denoted by  $\mu_R^n$ ) and  $\mu_R^1 = E(R - \mu_R) = 0$ , we obtain:

$$E(U(R)) = a_0 + a_2 \mu_R^2 + a_1 \mu_R^3 + \sum_{n=4}^{\infty} a_n \mu_R^n \quad (6)$$

S2. To obtain an approximation to  $E(U(R))$ , discard all terms involving moments higher than 2 or 3, obtaining equations (6a) and (6b):

$$E(U(R)) = a_0 + a_2 \mu_R^2 = a_0 + a_2 \sigma_R^2 \quad (6a)$$

$$E(U(R)) = a_0 + a_2 \mu_R^2 + a_3 \mu_R^3 = a_0 + a_2 \sigma_R^2 + a_3 \mu_R^3 \quad (6b)$$

Equation (6a) describes an investor whose expected utility analysis is based only on the mean ( $a_0 = U(\mu_R)$ ) and variance. The condition  $U'' < 0$  is usually imposed. This will force  $a_2 < 0$ , and make the investor risk averse.

Equation (6b) describes an investor who also is influenced by the third moment and thus considers skewness. Strict consistency of preference (Scott-Horvath [19]) will force  $a_3 > 0$ , in which case the investor is considered to have a positive skewness preference.

Although these approximations in (6a) and (6b) may give reasonable results under various conditions concerning the decrease or disappearance

of coefficients or moments, it should be clear that each approximation is subject to error in situations where the discarded tail of the series is large. The following are among the problems that may arise in the use of the above steps even in the simple case in which the return distribution is discrete and finite and the utility function  $U$  is analytic.

B. Some Problems Identified in the Taylor Series Approximation

PR1. The series expansion represents  $U(R)$  only over the region of convergence of the series, while the distribution of  $R$  may extend beyond this region of convergence. Consider, for example,  $U(R) = \ln(R)$  with  $\mu_R = 1.2$  (a relativistic value for options). The series obtained for

$\ln(R)$ ,  $\sum_{n=0}^{\infty} a_n (R-1.2)^n$ , diverges outside the interval  $(0, 2.4)$ . Thus an

$R$  of 5 (quite possible for options) cannot be substituted into the series-- the full series  $\sum a_n (5-1.2)^n$  diverges. PR1 says that equations (1)-(4) are not valid over the full range of relative returns.

PR2. Taking expectations of both sides of (4) will not be valid if the right side of (4) does not converge for some actual values of the random variable,  $R$ . This problem states that equations (5) and (6) cannot be derived with validity under the circumstances described in PR1.

PR3. Even if problems PR1 and PR2 are not encountered, it is possible that the discarding of higher order terms in (6) may lead to so much error that (6a) and/or (6b) are seriously inaccurate. Examples in which this is the case are given below.

## II. Illustration of Problems PR1-PR3 for a Stylized Utility Function

The above problems are best understood if discussed in the context of specific stylized examples. The simplicity of these examples should indicate that some caution is required in the general use of series approximations.

Consider the case in which  $U(R) = \ln(R)$  and  $\mu_R = 1.2$  (approximately the 6 month mean holding period return determined for out-of-the money options [15]).<sup>2</sup>

Figure 1 displays graphs of  $\ln(R)$  and the quadratic and cubic approximations obtained by using the first three (quadratic) or four (cubic) terms of the infinite series.

Note that the actual function  $\ln(R)$ , diverges substantially from the cubic or quadratic approximations for  $R \geq 2.4$ , but that the approximations work well for analysis of risky assets which do not range too far from  $\mu_R = 1.2$ . For an investment in which  $R$  ranged from 1 to 1.4, the approximating curves would give much the same utilities as  $\ln(R)$  itself. However, it is also clear that for risky assets with large variance, problems will occur. If an asset return varies from  $R = 2$  to  $R = 5.2$ , for example, the cubic and quadratic functions will give significantly different approximations for some values of  $\ln(R)$ . Actual numerical values of  $\ln(R)$  and its cubic and quadratic approximations (with relative error) are given in Table I.

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Insert Table I

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The cubic approximation curve has an inflection point at  $R = 1.6$ . At this point an investor whose true utility function is  $\ln(R)$  is

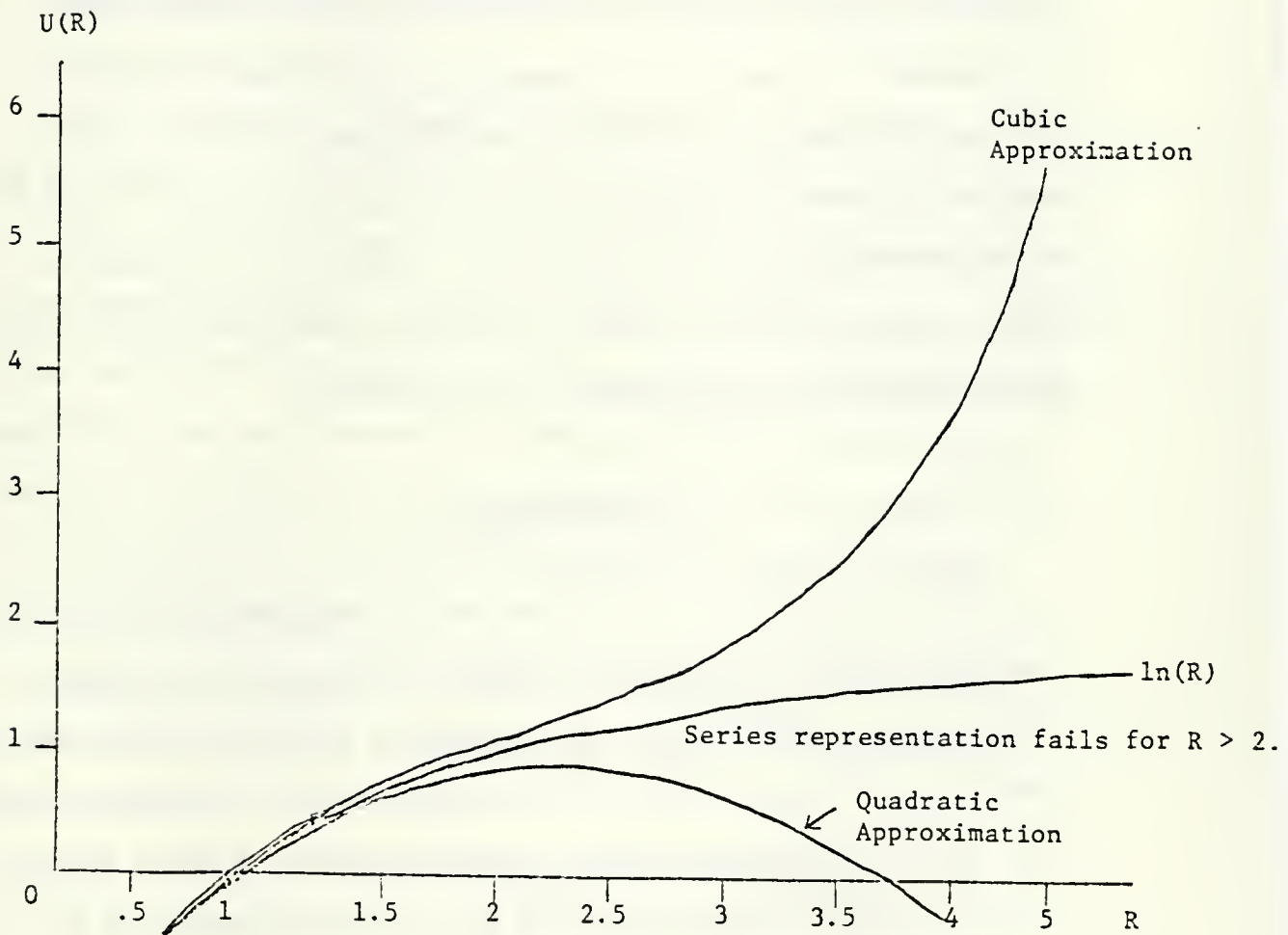


FIGURE 1. Graphs of  $\ln(R)$  and cubic and quadratic approximations

seriously misrepresented by use of the cubic approximation, since the cubic utility function implies that the investor is a risk lover for  $R > 1.6$ . The quadratic approximation attains a maximum at  $R = 2.4$ , and begins thereafter to seriously misrepresent the investor as having negative marginal utility of wealth.

PR2&3. If the utility of various returns,  $R$ , is seriously misrepresented by the use of series approximation, expected values obtained from those utilities may also be wildly incorrect (or correct only if the investment is restricted in range). This is illustrated by four different investments for which the actual values of  $E(\ln(R))$  and the cubic and quadratic approximations for  $E(\ln(R))$  are given in Table II. Note that restriction of the range of  $R$  to keep it in the region of Figure 1 where the curves closely approximate each other appears to be much more important than symmetry of distribution.

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Insert Table II

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Block IA in Table II illustrates the optimal case--restriction to a region of close approximation in Figure 1 along with symmetry of distribution. Block IB shows that symmetry of distribution does not prevent error. Block IC shows that a skewed distribution with restricted range may be well analyzed using a cubic approximation, while Block ID shows that highly skewed investments (e.g., options) should not be analyzed (for  $U(R) = \ln(R)$ ) by cubic approximation.<sup>3</sup>

In a sense, cubic and quadratic approximations actually change an investor from one whose utility function is  $U(R) = \ln(R)$  to one whose

utility function is a very different cubic or quadratic, as Table III illustrates. No matter what the shape of the original utility function, by discarding higher moments the investor's utility function is transformed into a quadratic or cubic.

Insert Table III

III. Applications to Recent Studies Assets with Skewed Distributions

A. Optimal portfolio size

Conine and Tamarkin [4] assert that preference for skewness will cause an investor to hold a limited rather than fully diversified portfolio under certain conditions. They use cubic approximations of  $E(U(R))$ , and their analysis assumes a homogeneous security universe. The following stylized example for a homogeneous securities universe satisfying their assumptions shows that the result of limited diversification may fail to hold for the actual utility function  $U(R) = \ln(R)$  while holding for the cubic approximation (and thus misleading the analyst). We first note the following theorem.

Theorem: In a securities universe in which all assets have the same mean return,  $E(\ln(R))$  is maximized by an asset A with 0 variance in its returns  $R_A$ .

Proof: Consider a risky asset with possible returns  $R_1, \dots, R_n$ , where the probability of  $R_i$  is  $p(R_i) = 1/n$ .

$$E(\ln(R)) = \frac{\ln(R_1) + \ln(R_2) + \dots + \ln(R_n)}{n} = \frac{\ln(R_1 \cdot R_2 \dots R_n)}{n}. \quad (7)$$

Since  $n$  is fixed, we need only maximize the product  $(R_1 \cdot R_2 \dots R_n)$  subject to the constraint  $R_1 + \dots + R_n = \text{a constant}$  (implied by identity of mean

return of all assets). By using the method of Lagrange multipliers, the solution  $R_1 = R_2 = \dots = R_n$  can be derived. (This easily can be generalized to a distribution of the form  $p(R_k) = p_k$ ,  $\sum p_k = 1$ . The distribution is given in a form similar to what would be obtained from historical return analysis). Equality of returns yields 0 variance.

Example. Consider a homogenous securities universe consisting of 5 securities each of which has the distribution:

$$p(R_i = 5.2) = .2$$

$$P(R_i = .2) = .8$$

The joint distribution is given using the notation:

$$\begin{aligned} P(R_1 = a_1, R_2 = a_2, R_3 = a_3, R_4 = a_4, R_5 = a_5) \\ = P(a_1, a_2, a_3, a_4, a_5) \end{aligned}$$

The joint distribution is:

$$\begin{aligned} P(5.2, .2, .2, .2, .2) &= .2 \\ P(.2, 5.2, .2, .2, .2) &= .2 \\ P(.2, .2, 5.2, .2, .2) &= .2 \\ P(.2, .2, .2, 5.2, .2) &= .2 \\ P(.2, .2, .2, .2, 5.2) &= .2 \\ P(\text{any other joint event}) &= 0 \end{aligned}$$

Under this distribution, in any given year exactly one security will have  $R = 5.2$ , and all others will have  $R = .2$ . A five year series of returns might appear as given in Table IV.

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Insert Table IV

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In this homogeneous universe, all securities have common mean,  $\mu_R = 1.2$ , variance,  $\sigma_R^2 = 4$ , and skewness,  $\mu_R^3 = 12$ . Covariances and coskewnesses are identical for distinct pairs of securities. According to the preceding theorem, an investor with utility function  $U(R) = \ln(R)$

should seek 0 variance in return. This can in fact be attained if one buys an equally weighted market portfolio, for which  $P(R = 1.2) = 1$  -- the investor can obtain a constant unvarying return of 1.2. Use of the cubic approximation to  $E(\ln(R))$  gives the misleading idea that the investor attaches higher utility to the risky venture of holding only one security, as Table V indicates. The quadratic approximation is inexact, but does identify the optimal utility portfolio.

Insert Table V

For the investor's actual utility function  $U = \ln(R)$ , complete diversification is optimal. Use of the cubic approximation makes the riskiest asset appear to be of highest expected utility, when in fact it is only of highest expected utility to a cubic risk lover. The approximation has treated the investor as a cubic risk lover.

B. Premium Calculation

Kane [8] uses series truncation to calculate the relative risk premium  $p$  for a risky asset, taking skewness into account. The premium is defined by the equation:

$$U(W(1+\bar{y}-p)) = E\{U[W(1+y)]\} \quad (8)$$

where  $W$  denotes initial wealth,  $y$  represents return on investment and  $\bar{y}$  the mean of  $y$ . Using a three moment approximation, the following formula is obtained.

$$p_m = \frac{1}{2} R(W) \sigma^2 - \frac{1}{6} S(W) E(y^3) \quad (9)$$

where:

$$S(W) = \frac{W^2 U''(W)}{U'(W)}, \quad R(W) = \frac{-W U''(W)}{U'(W)} \quad (10)$$

Kane is careful to state assumptions on the series. The following example only shows that his locally correct formula must be applied with the same care.

Example Consider an investment such that  $p(y=3) = 1/3$ ,  $p(y=0) = 2/3$ . Then  $\bar{y} = 1$ . If  $U = \ln(R)$  and  $W = 1$ , the equation for the true premium is:

$$\ln(2-p) = E(\ln(1+y)) = .462 \quad (11)$$

Solving equation (11) for  $p$  gives a value of .41. However, if the formula for  $p_m$  (equation (9)) with  $W = 1$ , is used, then:  $U(R) = \ln(R)$ ,  $S(W) = 2$ ,  $R(W) = 1$ ,  $\sigma^2 = 2$  and  $E(y^3) = 9$ , and one obtains  $p_m = -2$ . The approximate formula for premium is seriously in error when inappropriately used for a highly skewed asset. (Kane notes that plunging may become evident if the approximation is applied to highly skewed investments. This may be attributed to the fact that the three term Taylor series approximation to  $\ln(1+y)$  changes the utility function to one of a cubic risk lover).

#### C. Expected utility with a normal distribution of returns

One justification given (somewhat loosely) for use of only a quadratic approximation (MV) approach is that stock returns are nearly normally distributed and the normal distribution is determined by  $\mu_R$  and  $\sigma_R^2$ . However, as shown in equation (12) the full series for  $E(U(R))$  clearly requires more moments than  $\sigma_R^2$ , even when one drops odd moment terms which are 0:

$$E(U(R)) = U(\mu_R) - \frac{U^{(2)}(\mu_R)\sigma_R^2}{2!} - \left( \sum_{n=2}^{\infty} \frac{U^{(2n)}(\mu_R)}{(2n)!} \mu_R^{2n} \right) \quad (12)$$

The MV formula for  $E(U(R))$  contains only the first two terms--a potentially serious truncation. However, it is true that a qualitative description of investor behavior based on expected utility can be validly derived using only the first two terms. This occurs because the higher moments  $\mu_R^{2n}$  are related to  $\sigma_R^2$  as shown in equation (13).

$$\mu_R^{2n} = \frac{(2n)!(\sigma_R^2)^n}{(2^n)n!} \quad (13)$$

If  $U(R)$  is any utility function displaying strict consistency of preference, all derivatives  $U^{(2n)}$  will be  $\geq 0$  (Scott-Horvath [19]). Thus the investor is truly averse to all higher moments. However, if the investor lowers variance,  $\sigma_R^2$ , this will also diminish all higher moments with  $\sigma_R^2$ . The aversion to  $\sigma_R^2$  alone implied by the MV approach actually forces the aversion to all higher moments required by the full series, so a correct qualitative description of investor behavior is obtained using the MV approximation. However, the MV approximation to  $E(U(R))$  will overestimate the actual quantitative value of  $E(U(R))$ . It is not inconceivable that this could create problems in a theoretical calculation such as the risk premium calculation just completed.

### III. Conclusions

The broadening spectrum of financial securities which allow investors to create almost any desired expected return distribution motivates the study of the effect of higher distribution moments on investor utility. The purpose of this article is to emphasize certain precautions which

must be observed when solving for expected utility using a two or three moment Taylor series approximation.

Three problems were identified if a Taylor series is used to solve for expected utility. First, returns with high variability may fall outside the region of convergence of the Taylor series. Examples presented indicate that the analysis may be applied to skewed return distributions having return values in the region of convergence, but may be incorrect for normal distributions with large variance in returns. Second, series for utility functions with an infinite series of derivatives may not converge as desired. Discarding the terms beyond the second or third derivatives actually transforms the assumed utility function to one which is cubic or quadratic, a perhaps undesirable consequence. Finally, because the higher moments of non-symmetric return distributions exist and are non-zero, the use of a truncated approximation may seriously misstate expected utility.

The problems outlined above imply that optimization techniques presented in recent articles may provide misleading results when asset return distributions are non-symmetric. Future research should explore other means for incorporating higher moments into the preference structure for risky assets.

Footnotes

<sup>1</sup>Utility functions which exhibit necessary and sufficient conditions for portfolio analysis--separation, myopia [3,14] and Pareto Optimality of linear sharing rules [16] include: the (1) exponential  $U(R) = 1 - e^{-aR}$  ( $a > 0$ ), (2) logarithmic  $U(R) = \ln(R)$  and (3) power  $U(R) = R^b$  ( $0 < b < 1$ ).

<sup>2</sup>The logarithmic and power utility functions are most likely to create problems in expected utility analysis. Approximation of the exponential utility function by a truncated Taylor series is generally reliable.

<sup>3</sup>Table 2 indicates that Loistl's [13] conclusion must be carefully interpreted in light of his specific examples. For example, Loistl states "the mean-variance approximation is not a good approximation of the expected value of utility at all; however, it is more exact than a Taylor series expansion including higher terms of any order". Contrary to this statement, Blocks IA and IC of Table 2 show cases in which the mean-variance approximation is nearly exact. As long as returns lie in the region of convergence, it can be shown that adding more terms will improve the Taylor series approximation. Loistl's results are based on returns beyond the limit of convergence.

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TABLE I

Actual and Approximate Values of  $U = \ln(R)^*$ 

R	R-1.2	Actual $\ln(R)$	Cubic Approximation	Cubic/ $\ln$	Quadratic Approximation	Quadratic/ $\ln$
1.2	0	.18232	.18232	1	.18232	1
1.6	.4	.47000	.47245	1.00519	.46010	.97893
2	.8	.69315	.72553	1.04672	.62677	.90423
2.4	1.2	.87547	1.01565	1.16013	.68232	.77938
2.8	1.6	1.02962	1.41689	1.37613	.62677	.60874
3.2	2	1.16315	2.00331	1.72231	.46010	.39556
3.6	2.4	1.28093	2.84899	2.22415	.18232	.14234
4	2.8	1.38629	4.02844	2.90559	-.20657	-.14901
4.4	3.2	1.48160	5.61442	3.78942	-.70657	-.47689
4.8	3.6	1.56862	7.68232	4.89752	-1.31768	-.84003
5.2	4	1.64866	10.30580	6.25101	-2.03990	-1.23373
5.6	4.4	1.72277	13.55890	7.87040	-2.87323	-1.66780

\*Tabular entries are computed via expansion about  $R = 1.2$  where

$$U = \ln(R) = a_0 + a_1(R-1.2) + a_2(R-1.2)^2 + a_3(R-1.2)^3$$

and

$$a_0 = \ln(1.2) = .18232$$

$$a_1 = 1/1.2 = .83333$$

$$a_2 = -1/2(1.2)^2 = .34722$$

$$a_3 = 1/3(1.2)^3 = .19290$$

TABLE II

Actual and Approximate Expected Utility for Stylized Investments

	Return Range	Symmetric	$\sigma_R^2$	$\mu_R^3$	E(U) = E(ln(R))			Error Comparison	
					Actual	Cubic Approx.	Quadratic Approx.	$\frac{\text{Cubic}}{\text{Actual}}$	$\frac{\text{Quadratic}}{\text{Actual}}$
A p(R=1.2)=.6 p(R=1.1)=.2 p(R=1.3)=.2	.2	yes	.004	0	.18093	.18093	.18093	1.00000	1.00000
B p(R=.4)=.2 p(R=1.2)=.6 p(R=2.0)=.2	1.6	yes	.256	0	.06476	.09343	.09343	1.44267	1.44267
C p(R=1.1)=.8 p(R=1.6)=.2	.5	no	.04	.012	.17025	.16843	.17075	1.00293	.98933
D p(R=.2)=.8 p(R=5.2)=.2	5	no	4	12	-.95782	1.10825	-1.20657	-1.15705	1.25970

TABLE III

Taylor Series and Approximation formulas for  
Expected Utility with  $\mu_R = 1.2^*$

	$U = f(R)$	$U = \text{quadratic}$ Approximation	$U = \text{cubic}$ Approximation
Taylor Series For $U(R)$	$a_0 + a_1(R-1.2)$ $+ a_2(R-1.2)^2 + a_3(R-1.2)^3$ $+ a_4(R-1.2)^4 + \dots$ $+ a_n(R-1.2)^n + \dots$	$a_0 + a_1(R-1.2)$ $+ a_2(R-1.2)^2$	$a_0 + a_1(R-1.2)$ $+ a_2(R-1.2)^2$ $+ a_3(R-1.2)^3$
<u><math>E(U(R))</math></u> (only valid if $R$ stays in region of convergence)	$a_0 + a_1\sigma_R^2 + a_2\sigma_R^4$ $+ a_3\mu_R^3 + a_4\mu_R^4$ $+ \dots + a_n\mu_R^n + \dots$	$a_0 + a_2\sigma_R^2$	$a_0 + a_2\sigma_R^2 + a_3\mu_R^3$
<u><math>E(U(R))</math></u> discard all moments but $\mu_R^0, \mu_R^1, \mu_R^2$	$a_0 + a_2\sigma_R^2$		
<u><math>E(U(R))</math></u> discard all moments but $\mu_R^0, \mu_R^1$ $\mu_R^2$ and $\mu_R^3$	$a_0 + a_2\sigma_R^2 + a_3\mu_R^3$		

\*The above holds for any utility function  $U = f(R)$  which can be expanded about  $\mu_R$ . The  $a_i$  are Taylor series coefficients for  $f(R)$ .

TABLE IV  
 Typical Five Year Return Sequence  
 Security

	1	2	3	4	5
1	5.2	.2	.2	.2	.2
2	.2	5.2	.2	.2	.2
3	.2	.2	5.2	.2	.2
4	.2	.2	.2	5.2	.2
5	.2	.2	.2	.2	5.2

TABLE V

Exact and Approximate Expected Utility of Return

Number of Securities in Portfolio	Return Distribution	$E(\ln(R))$ $= \sum p_i \ln(R_i)$ (Actual)	Cubic Approximation $a_0 + a_2 \sigma_R^2 + a_3 u_R^3$	Quadratic Approximation $a_0 + a_2 \sigma_R^2$	Comparison	
					<u>Cubic</u> Actual	<u>Quadrat</u> Actual
1	$p(5.2) = .2$ $p(.2) = .8$	-.95782	1.10825	-1.20657	-1.15705	1.2597
2	$p(2.7) = .4$ $p(.2) = .6$	-.56836	- .19384	- .33851	.34043	.5955
3	$p(1.86) = .6$ $p(.2) = .4$	-.29628	- .09203	- .04916	.34175	.1825
4	$p(1.45) = .8$ $p(.2) = .2$	-.02464	.05935	.09552	-2.40888	-3.8765
5	$p(1.2) = 1.0$	.18232	.18232	.18232	1	1









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